IMPLICIT DIFFERENTIATION EXPLAINED.

Normally, when we differentiate an expression we simply write the following:

If $y = 5x^{3}$ then $\frac{dy}{dx} = 15x^{2}$ We need to re-think of $\frac{dy}{dx}$ in a different way: we need to think of $\frac{d}{dx}$ () as meaning :

"Differentiate the x's in the brackets"

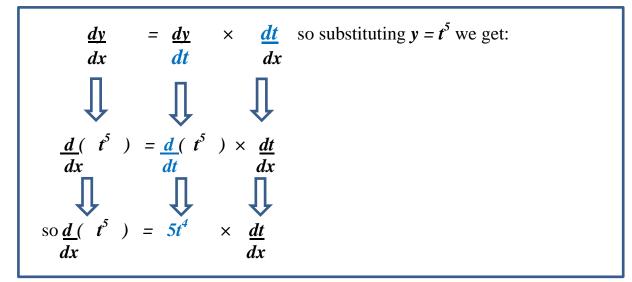
Sometimes this is referred to as: "Differentiate *with respect to x*" but it is perhaps more instructive just to think of it as "Differentiate the x's"

So $\frac{d}{dx}(x^3)$ means: "Differentiate the x's" and of course it equals $3x^2$ Similarly, $\frac{d}{dy}(y^4)$ means: "Differentiate the y's" and this equals $4y^3$ Also, $\frac{d}{dt}(t^6 + 5t + 3)$ means: "Differentiate the t's" and this equals $6t^5$ Now consider the expression:

$$\frac{d}{dx}(t^5)$$

This means to **differentiate the** x's but there are no x's, only t's!

Here we use a version of the **chain rule** normally stated as:



The DIFFERENCE between EXPLICIT and IMPLICIT Equations.

Firstly an $\underline{\mathbf{Ex}}$ plicit equation has y as the subject and just x's on the other side.

Eg $y = x^3 + x^2 - 4x + 7$

An \underline{Im} plicit equation has x's and y's mixed throughout the equation.

Eg $y^3 + x^4 + 5y^2 - 7x = 2$

It is often not possible to transform an **implicit** equation into an **explicit** equation so we use the method above to differentiate such equations.

Consider the equation:

$$y^3 + x^4 + 5y^2 - 7x = 2$$

If possible, it is quite a good idea to have x's on one side of the equation and y's on the other even though we cannot make y the single subject.

$$y^3 + 5y^2 = 2 + 7x - x^4$$

Now we differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}(y^3 + 5y^2) = \frac{d}{dx}(2 + 7x - x^4)$$
Remember the symbol $\frac{d}{dx}(y^3)$ means, differentiate the x's in the brackets.

We can do the right hand side (above) but the left hand side has y's **not** x's.

So using the chain rule idea explained earlier, we do the following:

Normally, we would just set out the answer as follows:

Find
$$\frac{dy}{dx}$$
 if $e^{3y} + \sin y = \ln(x) + x^4$
 $3e^{3y} \frac{dy}{dx} + \cos y \frac{dy}{dx} = \frac{1}{x} + 4x^3$
 $\frac{dy}{dx} (3e^{3y} + \cos y) = \frac{1}{x} + 4x^3$
 $\frac{dy}{dx} = \frac{1}{x} + 4x^3$
 $\frac{dy}{dx} = \frac{1}{x} + 4x^3$

EXTENSION.

Sometimes we need to use the **product rule** or **quotient rule** as well:

eg. Find y' if:

$$x^{3}y^{5} + xy = 9x$$

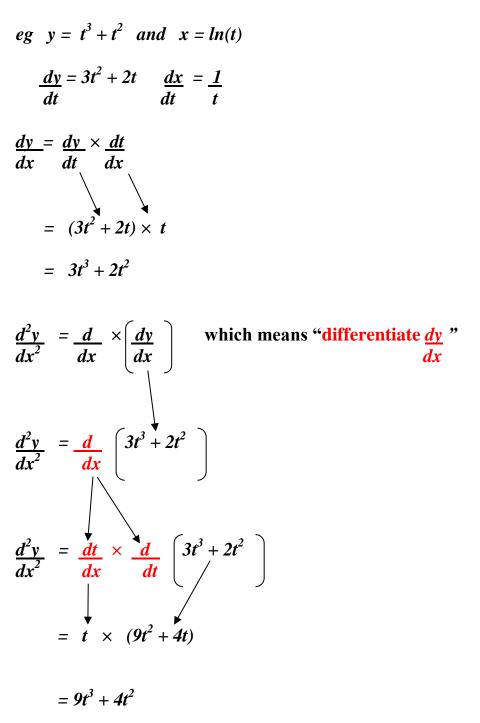
$$x^{3}5y^{4}y' + 3x^{2}y^{5} + xy' + 1y = 9$$

$$y'(x^{3}5y^{4} + x) = 9 - 3x^{2}y^{5} - y$$

$$y' = (9 - 3x^{2}y^{5} - y)$$

$$(x^{3}5y^{4} + x) = 9 - 3x^{2}y^{5} - y$$

A particularly difficult point is finding $\frac{d^2y}{dx^2}$ for **Parametric Equations**.



Finally, an interesting point is that we can find y'' in the following case in three ways: parametrically, implicitly and explicitly.

PARAMETRICALLY:

$$x = \sin t \qquad y = \cos t$$

$$\frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -\cos t$$

$$\frac{dy}{dt} = \frac{dy}{dt} \times \frac{dt}{dt} = -\frac{\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = -\frac{d}{dt}(\tan t) = -\frac{dt}{dx} \times \frac{d}{dt}(\tan t)$$

$$= -\frac{1}{\cos t} \times \sec^2 t = -\frac{1}{\cos^3 t} = -\frac{1}{y^3}$$

IMPLICITLY:

Consider
$$x^2 + y^2 = 1$$

So $2x + 2y y' = 0$
 $y' = -\underline{x}$
and $y'' = -\left(\frac{y \times 1 - x \times y'}{y^2}\right)$
 $= -\left(\frac{y + \underline{x}^2}{y^2}\right)$
 $= -\left(\frac{y^2 + x^2}{y^3}\right) = -\frac{1}{y^3}$

EXPLICITLY:

 $y = (1 - x^{2})^{\frac{1}{2}} (ignoring \pm)$ so $y' = \frac{1}{2} (1 - x^{2})^{-\frac{1}{2}} \times (-2x)$ $= \frac{-x}{(1 - x^{2})^{\frac{1}{2}}}$ And $y'' = -\left(\frac{(1 - x^{2})^{\frac{1}{2}} - x^{2}(1 - x^{2})^{-\frac{1}{2}}}{(1 - x^{2})}\right)$ $= -\left(\frac{(1 - x^{2})^{\frac{1}{2}} - x^{2}}{(1 - x^{2})^{\frac{1}{2}}}\right)$ $= -\left(\frac{(1 - x^{2})^{\frac{1}{2}} - x^{2}}{(1 - x^{2})^{\frac{1}{2}}}\right)$ $= -\left(\frac{(1 - x^{2}) - x^{2}}{(1 - x^{2})^{\frac{3}{2}}}\right)$ $= -\frac{1}{y^{3}}$

In your equation, for better understanding, it is probably better to change it to:

$$y^2 = 10 - x^2$$

Differentiating with respect to x:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\mathbf{y}^2 \right) = \frac{\mathrm{d}}{\mathrm{d}x} \left(10 - x^2 \right)$$

$$\frac{\mathbf{d}\mathbf{y}}{\mathbf{d}\mathbf{x}} \quad \frac{\mathbf{d}}{\mathbf{d}\mathbf{y}} \left[\mathbf{y}^2 \right] = \quad \frac{\mathbf{d}}{\mathbf{d}\mathbf{x}} \left[10 - \mathbf{x}^2 \right]$$

so we get:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{x}} \left[2\mathbf{y} \right] = -2\mathbf{x}$$

$$\frac{\mathbf{d}\mathbf{y}}{\mathbf{d}\mathbf{x}} = -\underline{\mathbf{x}}$$