INTRIGUING CONTINUED FRACTIONS.

I was thinking about a problem in an Eton Senior Mathematics Competition which asked to find the value in surd form of this continued fraction:

1.
$$\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6+\dots}}}}}$$

We can put $x = \dfrac{1}{6 + x}$

$$\text{so } x(6 + x) = 1$$

$$x^2 + 6x = 1$$

$$x^2 + 6x + 9 = 9 + 1$$

$$(x + 3)^2 = 10$$

so $x = -3 \pm \sqrt{10}$ (and neglecting the negative solution)

in this case $x = -3 + \sqrt{10}$

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2. The general case would be:

$$x = \cfrac{a}{b+\cfrac{a}{b+\cfrac{a}{b+\dots}}}$$

$$\text{so } x = \frac{a}{b + x}$$

giving us the quadratic equation :

$$x^2 + bx - a = 0$$

We <u>SHOULD</u> be able to make an equation like this to have IRRATIONAL or even UNREAL SOLUTIONS!

3. If we choose $a = 15$ and $b = 2$

$$x = \cfrac{15}{2 + \cfrac{15}{2 + \cfrac{15}{2 + \dots}}}$$

we get : $$x = \frac{15}{2 + x}$$

so $x^2 + 2x - 15 = 0$

and $(x - 3)(x + 5) = 0$

so $x = 3$ *and obviously not* -5

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4. Now here is an interesting thought, if we choose $a = -2$ and $b = 2$

$$x = \cfrac{-2}{2 - \cfrac{2}{2 - \cfrac{2}{2 - \dots\dots}}}$$

we get the equation : $$x = \frac{-2}{2 + x}$$

producing $x^2 + 2x + 2 = 0$

$$x^2 + 2x = -2$$

$$x^2 + 2x + 1 = 1 - 2$$

$$(x + 1)^2 = -1$$

so $x = -1 + i$ *or* $-1 - i$

(It is not quite so obvious whether to neglect one of these solutions.)

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5. Now consider the case where ***a = 2 and b = 0***

$$x = \cfrac{2}{0 + \cfrac{2}{0 + \cfrac{2}{0 + \cfrac{2}{0 + \ldots}}}}$$

or $x = 2 \div (2 \div (2 \div (2 \div (2 \div (\ldots$

so $x = \dfrac{2}{x}$

leading to $x^2 = 2$ ***and*** $x = \pm 2$

so $x = \pm\sqrt{2}$ ***I suppose we say*** $x = +\sqrt{2}$

Note: I think alarm bells should start to ring here because if we had a finite **EVEN** number of 2's such as 2÷(2÷(2÷(2))) it equals 1
but if we had a finite ODD number of 2's such as 2÷(2÷(2)) it equals 2.

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6. An even more alarming case is when ***a*** = ***– 1 and b = 0*** producing:

$$x = \cfrac{-1}{0 - \cfrac{1}{0 - \cfrac{1}{0 - \cfrac{1}{0 - \ldots}}}}$$

or $x = -1 \div (-1 \div (-1 \div (-1 \div (-1 \div (\ldots$

so $x = \dfrac{-1}{x}$

leading to $x^2 = -1$ ***and*** $x = \pm i$
and $x = i$ ***or*** $-i$

HOW ON EARTH CAN AN ARITHMETIC PROCESS INVOLVING REAL NUMBERS WITH NO SQUARE ROOT PROCESS BECOME A COMPLEX NUMBER?

Note:
If we had a finite even number such as -1÷(-1÷(-1÷(-1))) it equals +1
If we had a finite odd number such as -1÷(-1÷(-1)) it equals -1

……………………………….most intriguing!!!

CONTINUED FRACTIONS CONTINUED! A REVELATION!!!!!!!

If INFINITE continued fractions are defined as the limit of FINITE continued fractions:

i.e. $$x = \cfrac{a}{b + \cfrac{a}{b + \cfrac{a}{b + \dots}}}$$

then $$x = \lim_{n\to\infty} (c_n)$$

where $c_1 = \dfrac{a}{b}$ $\quad c_2 = \cfrac{a}{b + \cfrac{a}{b}}$ $\quad c_3 = \cfrac{a}{b + \cfrac{a}{b + \cfrac{a}{b}}}$

….. then clearly this requires that $b \neq 0$ (from the equation for c_1)

So this explains why, in questions 5 and 6 above, the algebraic method:

$$x = \cfrac{a}{b + \cfrac{a}{b + \cfrac{a}{b + \dots}}}$$

so $$x = \frac{a}{b + x}$$

giving us the quadratic equation :

$$x^2 + bx - a = 0$$

….is not correct because THE LIMITS DO NOT EXIST.