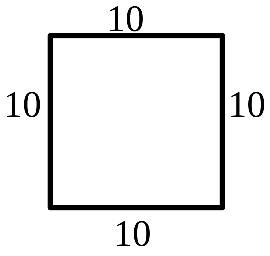
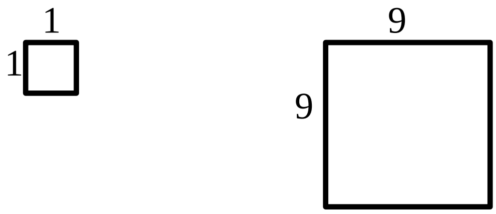
SUPERB INVESTIGATION SUITABLE AT MANY LEVELS.
(Teacher's guide)

1. Consider a piece of string 40 cm long.

   (a) If we make it into a square its area is obviously $10\times10 = 100$ cm^2



(b) If we cut the string into two pieces 4 cm and 36 cm and form two squares find the total area.



Total area = 1 + 81 = 82 cm^2

(c) If we cut the string into two pieces 8 cm and 32 cm and form two squares find the total area.

Total area = $2\times2 + 8\times8 = 68$ cm^2

(d) Now cut the string into pieces 12 cm and 28 cm.

Total area = $3\times3 + 7\times7 = 58$ cm^2

(e) Make a table of values

1st piece	2nd piece	Area 1st square	Area 2nd square	TOTAL AREA
4	36	1	81	82
8	32	4	64	68
12	28	9	49	58
16	24	16	36	52
20	20	25	25	50
24	16	36	16	52
28	12	49	9	58
32	8	64	4	68
36	4	81	1	82

(f) If the first piece is length ***x***, the other length is ***40 – x***

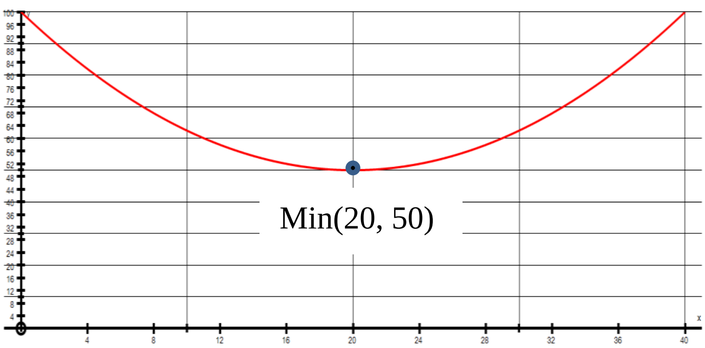
Find a formula for the total area:

$$A = \left(\frac{x}{4}\right)^2 + \left(\frac{(40-x)}{4}\right)^2$$

$$= \frac{x^2 + 1600 - 80x + x^2}{16}$$

$$= \frac{2x^2 - 80x + 1600}{16} \quad \text{or} \quad \frac{x^2}{8} - 5x + 100$$

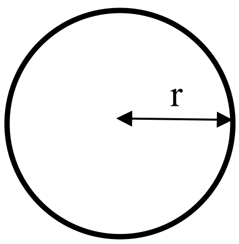
Draw a graph of this (possibly using Autograph or a Graphics Calculator)



Point out that this is a parabola. (The "right-way- up" type, so it has a minimum value not a maximum.)
Notice in particular that the minimum value is A = 50 when ***x*** = 20.

2. Consider a piece of string 40 cm long.(*HARDER*)

(a) If we make it into a CIRCLE we need to find the RADIUS, in order to calculate its area.



Students need to know $C = 2\pi r$

so that $2\pi r = 40$

$\pi r = 20$

$r = \frac{20}{\pi} \approx 6.366$

Area $= \pi r^2 = \pi \times (6.366)^2 \approx 127.3 \text{ cm}^2$

(b) If we cut the string into two pieces 4 cm and 36 cm and form two circles find the total area.

Radius of 1st circle $= \frac{2}{\pi} \approx 0.6366$

Area of 1st circle $= \pi \times (0.6366)^2$

$= 1.273 \text{ cm}^2$

Radius of 2nd circle $= \frac{18}{\pi} \approx 5.73$

Area of 2nd circle $= \pi \times (5.73)^2$

$= 103.1 \text{ cm}^2$

Total area $= 104.4 \text{ cm}^2$

(c) If the first piece is length ***x***, the other length is ***(40 – x)***

Find a formula for the total area: *(Quite complicated for students!)*

1st Circle

Circumference $= x = 2\pi r$

so $r = \dfrac{x}{2\pi}$

Area $= \pi r^2 = \pi \times \dfrac{x^2}{4\pi^2} = \dfrac{x^2}{4\pi}$

2nd Circle

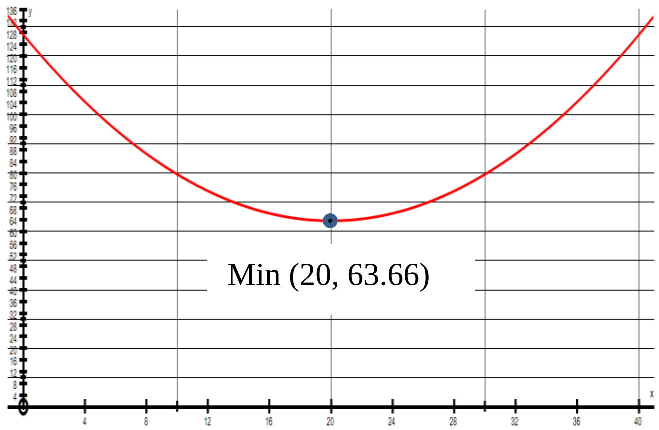
Circumference $= 40 - x = 2\pi r$

so $r = \dfrac{40 - x}{2\pi}$

Area $= \pi r^2 = \pi \times \dfrac{(40 - x)^2}{4\pi^2} = \dfrac{(40 - x)^2}{4\pi}$

Total Area = $\dfrac{x^2}{4\pi} + \dfrac{(40 - x)^2}{4\pi}$

If we draw this graph we do not have to go through lots of ***x*** values to get to the minimum area.



Point out that this is a parabola. (The right way up type, so it has a minimum value not a maximum.)
Notice in particular that the minimum value is A = 63.66 when ***x*** = 20.

3. ***(This is where this investigation has been leading.)***

A 40 cm piece of string is cut into 2 pieces.

One piece is made into a CIRCLE and the other is made into a SQUARE.

Find the minimum AREA.

Let one piece be of length ***x*** and the other is ***(40 – x)***

Let the 1st piece be made into a circle:

Circumference $= x = 2\pi r$

so $r = \dfrac{x}{2\pi}$

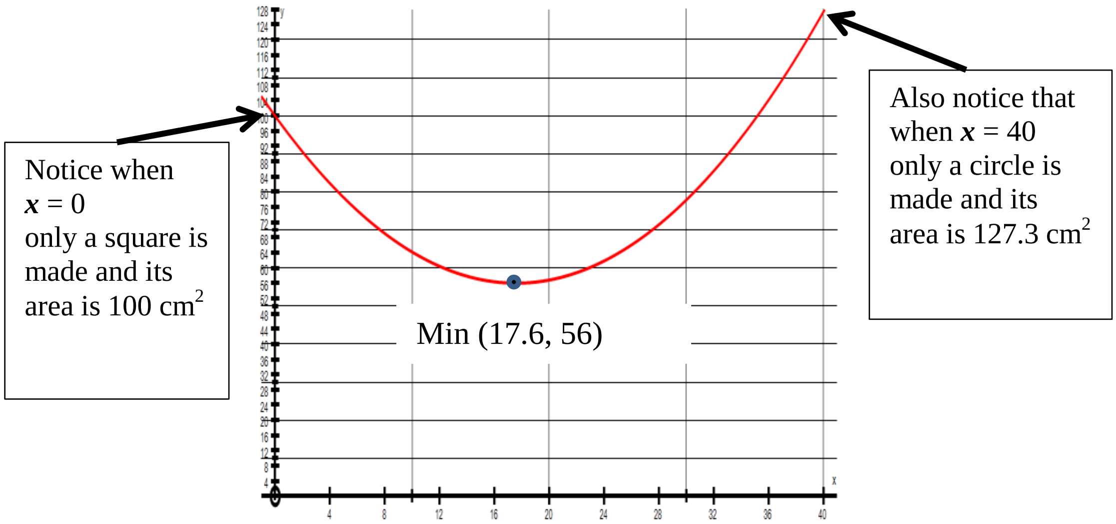
Area $= \pi r^2 = \pi \times \dfrac{x^2}{4\pi^2} = \dfrac{x^2}{4\pi}$

Let the 2nd piece be made into the square:

Each side will be $\dfrac{40-x}{4}$

So the area is $\dfrac{(40-x)^2}{16}$

Now we will draw the graph Area $y = \dfrac{x^2}{4\pi} + \dfrac{(40-x)^2}{16}$



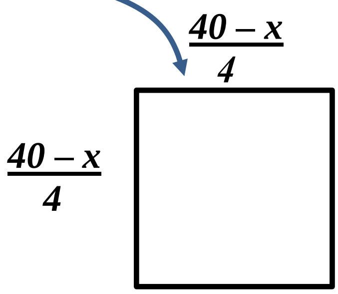
The minimum Area is 56.01 cm^2 when x = 17.6 cm

Although we have used a graph program to find this minimum value, this makes a good calculus problem.

Lengths are x and (40 – x). If circumference = x then $2\pi r = x$ ***so*** $r = \dfrac{x}{(2\pi)}$

Total Area $= \pi r^2 + \dfrac{(40-x)^2}{16}$

$$AREA = \frac{\pi x^2}{4\pi^2} + \frac{1600 - 80x + x^2}{16} = \frac{x^2}{4\pi} + 100 - 5x + \frac{x^2}{16}$$



$$\frac{d(Area)}{dx} = \frac{x}{2\pi} - 5 + \frac{x}{8} = 0 \text{ for max Area}$$

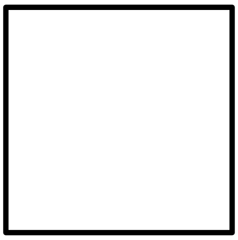
$$x\left(\frac{1}{2\pi} + \frac{1}{8}\right) = 5$$

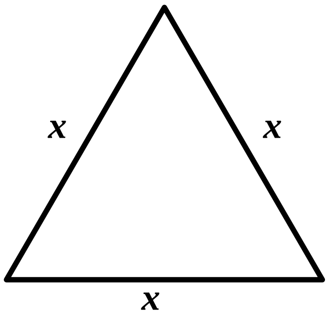
$$x \times 0.28415 = 5$$

$$x = 17.6 \text{ cm so } r = 2.8 \text{ cm}$$

So Min Area $= \pi \times 2.8^2 + 5.6^2 = 56$ ***cm***^{2}

A nice extension to this would be splitting the string into 2 pieces, making a SQUARE with one piece and an EQUILATERAL TRIANGLE with the other.





If we let each side of the triangle be x then the area could simply be written as $\frac{x \times x \times \sin 60}{2} = \frac{x^2 \sqrt{3}}{4}$

This leaves $(40 - 3x)$ left for the square so each side = $\frac{40 - 3x}{4}$

The area of the square = $\frac{(40 - 3x)^2}{16} = \frac{1600 - 240x + 9x^2}{16}$

The total area is A = $\frac{x^2 \sqrt{3}}{4} + \frac{1600 - 240x + 9x^2}{16}$

$$\frac{dA}{dx} = \frac{x\sqrt{3}}{2} - 15 + \frac{9x}{8} = 0 \text{ at min area}$$

$$x\left(\frac{\sqrt{3}}{2} + \frac{9}{8}\right) = 15$$

$$x \approx 7.53$$

Min A ≈ 43.5 cm^2