PRIMARY VALUES OF INDICES.

We know that $\sqrt[3]{8}$ *or* $8^{\frac{1}{3}} = 2$

Did YOU know that $\sqrt[3]{-8}$ *or* $(-8)^{\frac{1}{3}} \neq -2$

We say that $x^2 = 9$ has two solutions namely 3 and –3
But we also say $\sqrt{9} = 3$ but NOT $\sqrt{9} = -3$

Similarly, the equation $x^4 = 1$ has FOUR solutions, namely 1, i, –1 and –i
But we say that $\sqrt[4]{1} = 1$ but NOT $\sqrt[4]{1} = 1, i, -1$ and –i

When we find $\sqrt[2]{x}$ or $\sqrt[3]{x}$ or $\sqrt[4]{x}$ or $\sqrt[5]{x}$ etc. there is only ONE answer for each root and it is called the PRIMARY ROOT which is the 1st root found when solving $x^n = b$ using De Moivres Theorem.

Consider the equation $x^3 = 8$ which we know has 3 solutions not just the obvious solution $x = 2$
If we use De Moivre's Theorem to solve this we proceed as follows:

$$x^3 = 8$$
$$(r\,cis(\theta))^3 = 8$$
$$r^3\,cis(3\theta) = 8cis(0 + 360n)$$
$$r^3 = 8 \quad \text{and} \quad 3\theta = 360n$$
$$r = 2 \quad \text{and} \quad \theta = 0 + 120n = 0^0, 120^0, 240^0$$

$$x_1 = 2cis(0) = 2$$
$$x_2 = 2cis(120) = -1 + i\sqrt{3}$$
$$x_3 = 2cis(240) = -1 - i\sqrt{3}$$

I will refer to x_1 as the PRIMARY SOLUTION.
The other 2 solutions are generated from this solution by adding multiples of 120^0 to the "argument".

So we say that $\sqrt[3]{8}$ *or* $8^{\frac{1}{3}} = 2$

Now consider $x^3 = -8$

It "seems" we can just say $x = -2$ because $(-2)^3 = -8$ but -2 is not the Primary Solution!

Using De Moivre's theorem again:

$$x^3 = -8$$

$(r\,cis(\theta))^3 = -8$

$r^3\,cis(3\theta) = 8cis(180 + 360n)$

$r^3 = 8$ and $3\theta = 180 + 360n$

$r = 2$ and $\theta = 60 + 120n = 60^0, 180^0, 300^0$

$x_1 = 2cis(60) = 1 + i\sqrt{3}$

$x_2 = 2cis(180) = -2$

$x_3 = 2cis(240) = 1 - i\sqrt{3}$

The Primary Solution is $x_1 = 1 + i\sqrt{3} \approx 1 + 1.732i$

So $\sqrt[3]{-8}$ or $(-8)^{\frac{1}{3}} = 1 + 1.732i$ and NOT -2!

The other 2 solutions are generated from this solution by adding multiples of 120^0 to the "argument".

NB If we type $\sqrt[3]{-8}$ or $(-8)^{\frac{1}{3}}$ on the graphics calculator we get ***1 + 1.732i*** and not –2

Similarly, let us consider $x^4 = 1$

$(r\,cis(\theta))^4 = 1$

$r^4\,cis(4\theta) = 1cis(0 + 360n)$

$r^4 = 1$ and $4\theta = 360n$

$r = 1$ and $\theta = 0^0, 90^0, 180^0, 270^0$

$x_1 = cis(0) = 1$

$x_2 = cis(90) = i$

$x_3 = cis(180) = -1$

$x_4 = cis(270) = -i$

The Primary Solution is $x_1 = 1$

So that $\sqrt[4]{1}$ or $1^{\frac{1}{4}} = 1$

The other 3 solutions are generated from this solution by adding multiples of 90^0 to the "argument".

Compare this with $x^4 = -1$

$(r\,cis(\theta))^4 = -1$
$r^4\,cis(4\theta) = 1cis(180 + 360n)$
$r^4 = 1$ and $4\theta = 180 + 360n$
$r = 1$ and $\theta = 45 + 90n = 45^0, 135^0, 225^0, 315^0$

$x_1 = cis(0) = cos45 + isin45 = 0.707 + i0.707$
$x_2 = cis(90) = cos135 + isin135 = -0.707 + i0.707$
$x_3 = cis(180) = cos225 + isin225 = -0.707 - i0.707$
$x_4 = cis(270) = cos315 + isin315 = 0.707 - i0.707$

Notice that none of these solutions is a real number!

The Primary Solution is $x_1 = 0.707 + i0.707$

The other 3 solutions are generated from this solution by adding multiples of 90^0 to the "argument".

So that $\sqrt[4]{-1}$ *or* $(-1)^{\frac{1}{4}} = 0.707 + i0.707$ *which is verified by the graphics calculator.*

Consider $x^5 = 32$

$(r\,cis(\theta))^5 = 32$
$r^5\,cis(5\theta) = 32cis(0 + 360n)$
$r^5 = 32$ and $5\theta = 360n$
$r = 2$ and $\theta = 72n = 0^0, 72^0, 144^0, 216^0, 288^0$

$x_1 = 2cis(0) = 2cos0 + 2isin0 = 2$
$x_2 = 2cis(72) = 2cos72 + 2isin72 = 0.62 + 1.9i$
$x_3 = 2cis(144) = 2cos144 + 2isin144 = -1.62 + 1.18i$
$x_4 = 2cis(216) = 2cos216 + 2isin\ 216 = -1.62 - 1.18$
$x_5 = 2cis\ (288) = 2cos288 + 2isin288 = 0.61 - 1.9i$

The Primary Solution is $x_1 = 2$

The other 4 solutions are generated from this solution by adding multiples of 72^0 to the "argument".

So that $\sqrt[5]{32}$ *or* $32^{\frac{1}{5}} = 2$ *which is verified by the graphics calculator.*

Consider $x^5 = -32$

$(\ r\ cis(\theta)\ )^5 = -32$
$r^5\ cis(5\theta) = 32cis(180 + 360n)$
$r^5 = 32$ and $5\theta = 180 + 360n$
$r = 2$ and $\theta = 36 + 72n = 36^0, 108^0, 180^0, 252^0, 324^0$

$x_1 = 2cis(36) = 2cos36 + 2isin36 = 1.62 + 1.18i$
$x_2 = 2cis(108) = 2cos108 + 2isin108 = -0.62 + 1.9i$
$x_3 = 2cis(180) = 2cos180 + 2isin180 = -2$
$x_4 = 2cis(252) = 2cos252 + 2isin\ 252 = -0.62 - 1.9$
$x_5 = 2cis\ (324) = 2cos324 + 2isin324 = 1.62 - 1.18i$

The Primary Solution is $x_1 = 1.62 + 1.18i$

The other 4 solutions are generated from this solution by adding multiples of 72^0 to the "argument".

So that $\sqrt[5]{-32}$ or $(-32)^{\frac{1}{5}} = 1.62 + 1.18i$ which is verified by the graphics calculator.

If we go right back to $x^2 = 9$

$(r\,cis(\theta))^2 = 9$
$r^2\,cis(2\theta) = 9cis(0 + 360n)$

$r^2 = 9$ and $2\theta = 360n$
$r = 3$ and $\theta = 0^0, 180^0$

$x_1 = 3cis(0) = 3$
$x_2 = 3cis(180) = -3$

so √9 = 3 because it is the primary solution,
not because it is "a positive number" nor any other reason.

The equation $y^5 = -32$ is not the same as $y = \sqrt[5]{-32}$

The equation $y^5 = -32$ has 5 solutions
but $y = \sqrt[5]{-32}$ only has 1 solution (the primary solution)

Similarly:
$y^2 = 9$ has 2 solutions $y = +3$ or -3
but $y = 9^{1/2}$ only has 1 solution $y = +3$