IMPLICIT DIFFERENTIATION EXPLAINED.

Normally, when we differentiate an expression we simply write the following:

If $y = 5x^3$

then $\frac{dy}{dx} = 15x^2$

We need to re-think of $\frac{dy}{dx}$ in a different way:

we need to think of $\frac{d}{dx}(\quad)$ as meaning :

"**Differentiate the *x*'s in the brackets**"

Sometimes this is referred to as: "Differentiate ***with respect to*** *x*" but it is perhaps more instructive just to think of it as "**Differentiate the *x*'s**"

So $\frac{d}{dx}(x^3)$ means: "**Differentiate the x's** " and of course it equals $\mathbf{3x^2}$

Similarly, $\frac{d}{dy}(y^4)$ means: "**Differentiate the y's** " and this equals $\mathbf{4y^3}$

Also, $\frac{d}{dt}(t^6 + 5t + 3)$ means: "**Differentiate the t's**" and this equals $\mathbf{6t^5}$

Now consider the expression:

$$\frac{d}{dx}\left(t^5\right)$$

This means to **differentiate the *x*'s but there are no *x*'s, only *t*'s!**

Here we use a version of the **chain rule** normally stated as:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$ so substituting $y = t^5$ we get:

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\frac{d}{dx}\left(t^5\right) = \frac{d}{dt}\left(t^5\right) \times \frac{dt}{dx}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

so $$\frac{d}{dx}\left(t^5\right) = 5t^4 \times \frac{dt}{dx}$$

The DIFFERENCE between EXPLICIT and IMPLICIT Equations.

Firstly an **Ex**plicit equation has ***y*** as the subject and just ***x***'s on the other side.

Eg $y = x^3 + x^2 - 4x + 7$

An **Im**plicit equation has ***x***'s and ***y***'s mixed throughout the equation.

Eg $y^3 + x^4 + 5y^2 - 7x = 2$

It is often not possible to transform an **implicit** equation into an **explicit** equation so we use the method above to differentiate such equations.

Consider the equation:

$$y^3 + x^4 + 5y^2 - 7x = 2$$

If possible, it is quite a good idea to have ***x***'s on one side of the equation and ***y***'s on the other even though we cannot make ***y*** the single subject.

$$y^3 + 5y^2 = 2 + 7x - x^4$$

Now we differentiate both sides of the equation *with respect to* ***x*.**

$$\frac{d}{dx}\left(y^3 + 5y^2 \right) = \frac{d}{dx}\left(2 + 7x - x^4 \right)$$

Remember the symbol $\frac{d}{dx}(\quad)$ means, differentiate the ***x***'s in the brackets.

We can do the right hand side (above) but the left hand side has ***y***'s **not *x***'s.

So using the chain rule idea explained earlier, we do the following:

$$\frac{dy}{dx} \times \frac{d}{dy}\left(y^3 + 5y^2 \right) = \frac{d}{dx}\left(2 + 7x - x^4 \right)$$

$$\frac{dy}{dx}\left(3y^2 + 10y \right) = 7 - 4x^3$$

$$\frac{dy}{dx} = \frac{(7 - 4x^3)}{(3y^2 + 10y)}$$

Normally, we would just set out the answer as follows:

Find $\frac{dy}{dx}$ *if* $e^{3y} + \sin y = \ln(x) + x^4$

$$3e^{3y}\frac{dy}{dx} + \cos y\,\frac{dy}{dx} = \frac{1}{x} + 4x^3$$

$$\frac{dy}{dx}\left(3e^{3y} + \cos y\right) = \frac{1}{x} + 4x^3$$

$$\frac{dy}{dx} = \frac{\frac{1}{x} + 4x^3}{(3e^{3y} + \cos y)}$$

EXTENSION.

Sometimes we need to use the **product rule** or **quotient rule** as well:

eg. Find y' if :

$$x^3y^5 + xy = 9x$$

$$x^3\,5y^4\,y' + 3x^2y^5 + x\,y' + 1\,y = 9$$

$$y'(x^3\,5y^4 + x) = 9 - 3x^2y^5 - y$$

$$y' = \frac{(9 - 3x^2y^5 - y)}{(x^3\,5y^4 + x)}$$

A particularly difficult point is finding $\frac{d^2y}{dx^2}$ for **Parametric Equations**.

eg $y = t^3 + t^2$ *and* $x = \ln(t)$

$$\frac{dy}{dt} = 3t^2 + 2t \qquad \frac{dx}{dt} = \frac{1}{t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= (3t^2 + 2t) \times t$$

$$= 3t^3 + 2t^2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \times \left(\frac{dy}{dx}\right)$$

which means "differentiate $\frac{dy}{dx}$ "

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3t^3 + 2t^2\right)$$

$$\frac{d^2y}{dx^2} = \frac{dt}{dx} \times \frac{d}{dt}\left(3t^3 + 2t^2\right)$$

$$= t \times (9t^2 + 4t)$$

$$= 9t^3 + 4t^2$$

Finally, an interesting point is that we can find $\boldsymbol{y''}$ in the following case in **three** ways: **parametrically, implicitly and explicitly.**

PARAMETRICALLY:

$$x = \sin t \qquad\qquad y = \cos t$$

$$\frac{dx}{dt} = \cos t \qquad\qquad \frac{dy}{dt} = -\cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}(\tan t) = -\frac{dt}{dx} \times \frac{d}{dt}(\tan t)$$

$$= -\frac{1}{\cos t} \times \sec^2 t = \frac{-1}{\cos^3 t} = -\frac{1}{y^3}$$

IMPLICITLY:

Consider $x^2 + y^2 = 1$

$$\text{So} \quad 2x + 2y\,y' = 0$$

$$y' = -\frac{x}{y}$$

$$\text{and} \quad y'' = -\left(\frac{y \times 1 - x \times y'}{y^2}\right)$$

$$= -\left(\frac{y + \frac{x^2}{y}}{y^2}\right)$$

$$= -\left(\frac{y^2 + x^2}{y^3}\right) = -\frac{1}{y^3}$$

EXPLICITLY:

$$y = (1 - x^2)^{1/2} \quad \text{(ignoring } \pm\text{)}$$

$$\text{so} \quad y' = \tfrac{1}{2}(1 - x^2)^{-1/2} \times (-2x)$$

$$= \frac{-x}{(1 - x^2)^{1/2}}$$

$$\text{And } y'' = -\left\{\frac{(1 - x^2)^{1/2} - x^2(1 - x^2)^{-1/2}}{(1 - x^2)}\right\}$$

$$= -\left\{\frac{(1 - x^2)^{1/2} - \dfrac{x^2}{(1 - x^2)^{1/2}}}{(1 - x^2)}\right\}$$

$$= -\left[\frac{(1 - x^2) - x^2}{(1 - x^2)^{3/2}}\right]$$

$$= -\frac{1}{y^3}$$

In your equation, for better understanding, it is probably better to change it to:

$$y^2 = 10 - x^2$$

Differentiating with respect to x:

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(10 - x^2\right)$$

$$\frac{dy}{dx}\frac{d}{dy}\left(y^2\right) = \frac{d}{dx}\left(10 - x^2\right)$$

so we get:

$$\frac{dy}{dx}\left(2y\right) = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$