**RECTANGULAR and POLAR FORM of Complex Numbers.**

1. imag imag

 ***i* P *i* P**

 √2

 1

 1 450

 O 1 real O 1 real

 ***z = 1 + i and z = √2 cis(450)***

 **The above are two ways to write the same complex number.**

If we examine ***z = √2 cis(450)*** we get ***z = √2 (cos450 + i sin450)***

 ***= √2 ( 1 + i )***

 ***√2 √2***

  ***= 1 + i***

Clearly the two forms of ***z*** are equal.

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2. Generally, consider the complex number represented by **P** below:

 imag

 **P**

 ***r***

 ***y***

 ***θ***

 ***O x*** real

The complex number represented by the point P on this Argand diagram can be written as:

***z = x + iy but using basic trigonometry: x = r cosθ and y = r sinθ***

 so simply substituting we get ***z = x + iy***

 ***=*** ***rcosθ + i rsinθ***

 ***= r (cosθ + i sinθ )***

 Which we write in the short form ***z = r cis (θ)***

IMPORTANT POINT: ***r*** is the **LENGTH** of OP so it is ALWAYS POSITIVE.



1

***Here z = 3 + 0i in rectangular form.***

***The length of z = 3***

***also called the modulus of z or │z │***

***The angle z makes with the positive real axis is 00***

***also called arg(z)***

***So in polar form z = 3 cis (00)***

real

imag

2

imag

***Here z = 0 + 3i in rectangular form.***

***The length of z or │z │ = 3 ( not 3i)***

***The angle z makes with the positive real axis is arg(z) = 900***

***So in polar form z = 3 cis (900)***



real

3

imag



***Here z = -3 + 0i in rectangular form.***

***The length of z or │z │ = 4 ( not -4)***

***The angle z makes with the positive real axis is arg(z) = 1800***

***So in polar form z = 4 cis (1800)***

real

imag

4

***Here z = 0 – 2i in rectangular form.***

***The length of z or │z │ = 2 ( not -2i)***

***The angle z makes with the positive real axis is arg(z) = 2700 or -900***

***So in polar form z = 2 cis (2700)***

 ***or z = 2 cis(-900)***



real



5

***Here z = 4 + 3i in rectangular form.***

***The length of z or │z │= √(42 + 32) = 5 by Pythagoras’ theorem.***

***The angle z makes with the positive real axis is tan-1 (¾) ≈36.90***

***so arg(z) ≈36.90***

***So in polar form z = 5 cis (36.90)***

real

imag

imag

6

***Here z = -3 + 3i in rectangular form.***

***The length of z or │z │ = √(32 + 32) = √18 by Pythagoras’ theorem.***

***The angle z makes with the positive real axis is arg(z) = 180 – 45 = 1350***

***So in polar form z = √18 cis (1350)***



θ = 1350

450

real

imag

7



450

θ = 2250

***Here z = -4 – 4i in rectangular form.***

***The length of z or │z │ = √(42 + 42)= √32***

***The angle z makes with the positive real axis is arg(z) = 2250 or -1350***

***So in polar form z = √32 cis (2250)***

 ***or z = √32 cis (-1350)***

real

imag

8

***Here z = 2 – 2i in rectangular form.***

***The length of z or │z │ = √8***

***The angle z makes with the positive real axis is arg(z) = 3150 or -450***

***So in polar form z = √8 cis (3150)***

 ***or z = √8 cis(-450)***



θ = 3150

-450

real

imag

9\*

***Here z = -5 + 3i in rectangular form.***

***The length of z or │z │ = √(25 + 9)***

 ***= √34***

***Angle α = tan-1(3/5) ≈ 310***

***So θ, the angle z makes with the positive real axis is arg(z) = 180 – 31 = 1490***

***So in polar form z = √34 cis (1490)***



θ

α

3

5

real

imag

10\*

***Here z = -5 – 2i in rectangular form.***

***The length of z or │z │ = √(25 + 4) = √29***

***Angle α = tan-1(2/5) ≈ 21.80***

***So θ, the angle z makes with the positive real axis is arg(z) = 180 +201.8 = 201.80***

***So in polar form z = √29 cis (201.80)***



θ

α

real

imag

11\*

***Here z = 1 – 3i in rectangular form.***

***The length of z or │z │ = √(9 + 1) = √10***

***Angle α = tan-1(3/1) ≈ 71.60***

***So θ, the angle z makes with the positive real axis is arg(z) = 360 – 71.6 = 288.40***

***So in polar form z = √10 cis (288.40)***



α

θ

real

**SPECIAL CASES.**

**Look out for the “special triangles”**

 **30**

 **2 √3 √2 1**

 **60**

 **1 1**

 **1 imag**



α

θ

2

√3

***Here z = -1 + i√3 in rectangular form.***

***The length of z or │z │***

***= √(3 + 1) = √4 = 2***

***α is clearly 600*** (see special triangle)

***or tan – 1(√3) = 600***

***So θ = 180 – 60 = 1200***

***So in polar form z = 2 cis 1200***

real

 **2 imag**



θ

1

 √3

2

***Here z = √3 + i in rectangular form.***

***The length of z or │z │***

***= √(3 + 1) = √4 = 2***

***θ is clearly 300*** (see special triangle)

***So in polar form z = 2 cis 300***

real

**SPECIAL NOTE:**

If we use a constant real number ***p***, such as ***z = p + 0i***

obviously ***p*** could be a **positive** number or a **negative** number.

This means that we cannot write ***z*** in its **polar form** unless

we know whether ***p*** is positive or negative.

If p is a **positive** real number then ***z*** looks like this:

and ***arg(z)*** would be 00

so ***z = p cis(00)***

But if p is a **negative** real number then ***z*** looks like this:

 1800

and ***arg(z)*** would be 00

so ***z = p cis(1800) not – p cis(1800)***

However, if we say ***p***  is any real number (positive or negative) then obviously we could say that even powers such as ***p2 or p4 or p6 etc*** would be POSITIVE and any odd powers such as ***p1 or p3 or p5 etc*** would be NEGATIVE.

Therefore if we want to express ***z = p2 + 0i*** in polar form

we can be confident that

***arg(z)*** would be 00

so ***z = p2 cis(00) length = p2***

And if we want to express ***z = – p2 + 0i*** in polar form

we can be confident that 1800

and ***arg(z)*** would be 1800

so ***z = p2 cis(1800) obviously not – p2cis(1800) length = +p2***

**In the following examples we will assume that the variable *p* is a POSITIVE REAL NUMBER.**

**FURTHER SPECIAL CASES:**

1. Suppose ***z = p + 0i imag***

 ***Clearly │z │= p***

 ***and arg(z) = 00***

 ***p real***

 ***so z = p cis(00)***

***\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_***

2. Suppose ***z = 0 + pi imag***

 ***Clearly │z │= p (NOT pi) p***

 ***and arg(z) = 900***

 ***real***

 ***so z = p cis(900)***

***\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_***

3. Suppose ***z = -p + 0i imag***

 ***Clearly │z │= length = p (NOT –p)***

 ***and arg(z) =1800***

 ***p real***

 ***so z = p cis(1800)***

***\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_***

4. Suppose ***z = 0 – pi imag***

 ***Clearly │z │= length = p (NOT –pi)***

 ***and arg(z) = 2700 or -900***

 ***real***

 ***so z = p cis(2700) p***

***\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_***

5. Suppose ***z = p + pi imag***

 ***Clearly │z │= √(p2 + p2) = √2p2 = p√2***

 ***and arg(z) = 45 r p***

 ***p real***

 ***so z = p√2 cis(450)***

6. Suppose ***z = -p + pi imag***

 ***Clearly │z │= √(p2 + p2) = √2p2 = p√2***

 ***and arg(z) = 1350 p r***

 ***p real***

 ***so z = p√2 cis(1350)***

***\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_***

7. Suppose ***z = -p – pi imag***

 ***Clearly │z │= √(p2 + p2) = √2p2 = p√2***

 ***and arg(z) = 2250***

 ***p real***

 ***so z = p√2 cis(2250) p r***

***\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_***

8\*. Suppose ***z = p +( p√3)i imag***

 ***Clearly │z │= r = √(p2 + 3p2) r p√3***

 ***= √ 4p2***

 ***= 2p 600***

 ***p real***

***and arg(z) = tan – 1  p√3 = 600***

 ***p***

 ***so z = 2p cis(600) This triangle should be recognised***

 ***as one of our special triangles.***

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 **Always look out for the “special triangles”**

 **30**

 **2 √3 √2 1**

 **60 45**

 **1 1**