The Vanishing Points of Inflection.

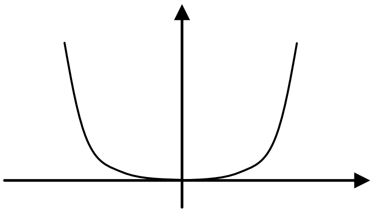
A point of inflection is when the gradient stops increasing and starts decreasing or vice versa.

Often, we can say that points of inflection occur when $y'' = 0$ **but not always**!

eg if $y = x^4$

$$y' = 4x^3$$

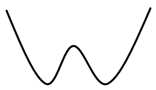
$$y'' = 12x^2 = 0 \text{ if } x = 0$$



But this graph has a minimum point at $x = 0$, not a point of inflection.

Consider the curve $y = x^4 - 4x^3 + 6x^2$

We would probably expect to get a typical curve such as :



but the gradient $y' = 4x^3 - 12x^2 + 12x = 4x(x^2 - 3x + 4)$

The gradient is zero if $x = 0$ because $x^2 - 3x + 4 \neq 0$ (only 1 turning point)

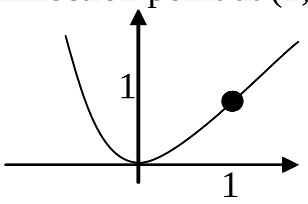
The 2nd derivative $y'' = 12x^2 - 24x + 12$

$$= 12(x^2 - 2x + 1)$$

$$= 12(x - 1)^2$$

This is zero if $x = 1$, so we would expect an inflection point at (1, 3)

The actual curve looks something like this:



<u>To solve this puzzle we need to look at curves with very similar equations</u>.

Consider the curve $\boldsymbol{y = x^4 - 4x^3 + 4x^2 = x^2(x^2 - 4x + 4) = x^2(x - 2)^2}$

The gradient $\boldsymbol{y' = 4x^3 - 12x^2 + 8x}$

$\boldsymbol{= 4x(x^2 - 3x + 2)}$

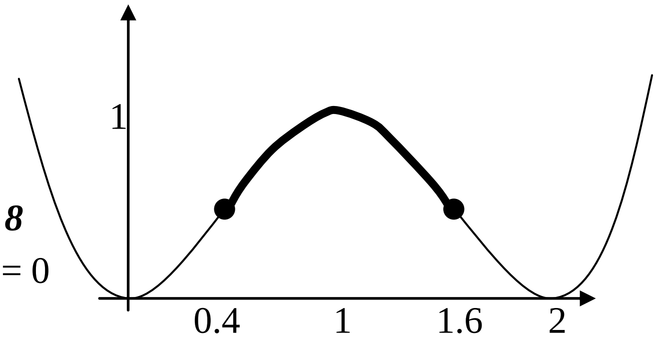
$\boldsymbol{= 4x(x - 1)(x - 2)}$

The gradient = 0 if $\boldsymbol{x = 0}$, 1 and 2

The 2nd derivative $\boldsymbol{y'' = 12x^2 - 24x + 8}$

The points of inflection are when $\boldsymbol{y''} = 0$

when $\boldsymbol{x} \approx 0.4$ and 1.6



Notice particularly that the curve is "concave down" between the two points of inflection in the interval $\boldsymbol{0.4 < x < 1.6}$

(ie This is the interval during which the gradient is decreasing)

Now consider the curve $\boldsymbol{y = x^4 - 4x^3 + 5x^2}$

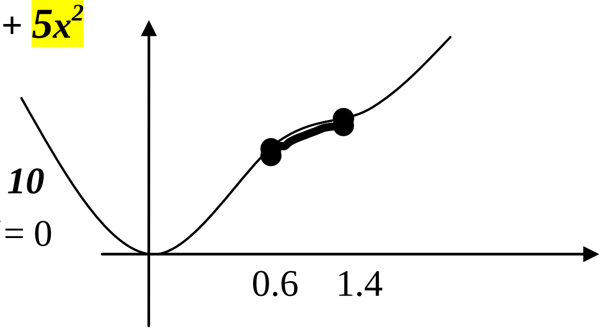
The gradient $\boldsymbol{y' = 4x^3 - 12x^2 + 10x}$

The gradient = 0 only if $\boldsymbol{x} = 0$.

The 2nd derivative $\boldsymbol{y'' = 12x^2 - 24x + 10}$

The points of inflection are when $\boldsymbol{y''} = 0$

when $\boldsymbol{x} \approx 0.6$ and 1.4



Notice particularly that the curve is "concave down" between the two points of inflection in the interval $\boldsymbol{0.6 < x < 1.4}$

(ie The gradient is decreasing in the small interval $\boldsymbol{0.6 < x < 1.4}$)

Similarly consider the curve $y = x^4 - 4x^3 + 5.8\,x^2$

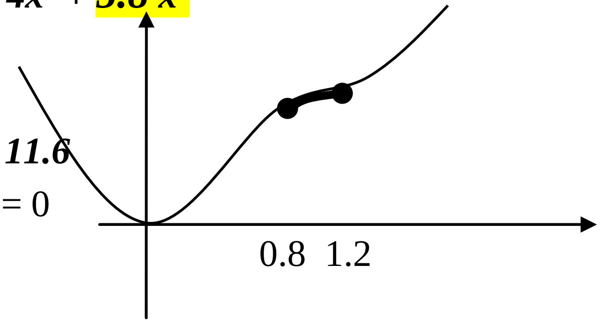
The gradient $y' = 4x^3 - 12x^2 + 11.6x$

The gradient = 0 only if $x = 0$.

The 2nd derivative $y'' = 12x^2 - 24x + 11.6$

The points of inflection are when $y'' = 0$

when $x \approx 0.8$ and 1.2



Notice particularly that the curve is "concave down" between the two points of inflection in the interval $0.8 < x < 1.2$

(ie The interval during which the gradient is decreasing, is getting smaller.)

Finally, reconsidering the curve $y = x^4 - 4x^3 + 6x^2$

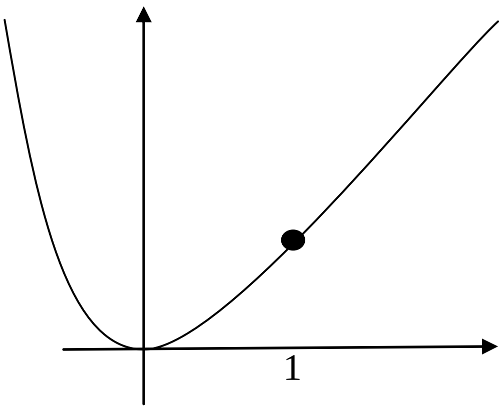
The 2nd derivative $y'' = 12x^2 - 24x + 12$

$$= 12(x^2 - 2x + 1)$$

$$= 12(x - 1)^2$$

We see that $y'' = 0$ only if $x = 1$

Normally, we would get two values of x but in this limiting case, the two values have converged to $x = 1$ so the curve has no interval in which it is concave down.



Although $y'' = 0$ at $x = 1$, the curve does not have a point of inflection because the gradient never decreases. It is increasing constantly as x increases.