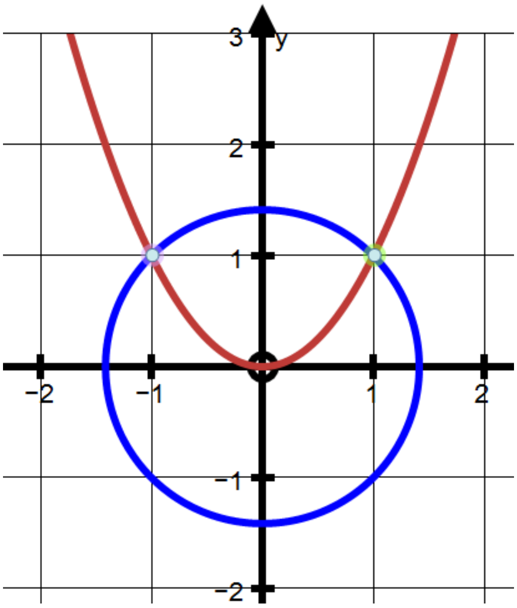
A simple intersection problem that turned out to be not so simple!

Find the intersection points of $y = x^2$ *and* $y^2 + x^2 = 2$

I decided to draw the graphs then look at the algebra…



It seems perfectly obvious and straightforward that the curves intersect at (1, 1) and (–1, 1) but the algebra tells a different story!

$$Subs\ y = x^2\ into\ y^2 + x^2 = 2$$

$$(x^2)^2 + x^2 = 2$$

$$x^4 + x^2 = 2$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2 - 1)(x^2 + 2) = 0$$

$$(x - 1)(x + 1)\left(x - i\sqrt{2}\right)\left(x + i\sqrt{2}\right) = 0$$

$$x = \pm 1\ and\ \pm\ i\sqrt{2}$$

The question is "How do I account for the two solutions: $= \pm\ i\sqrt{2}$?"

If $x = \pm i\sqrt{2}$ then $y = -2$ which is a **real y value**!

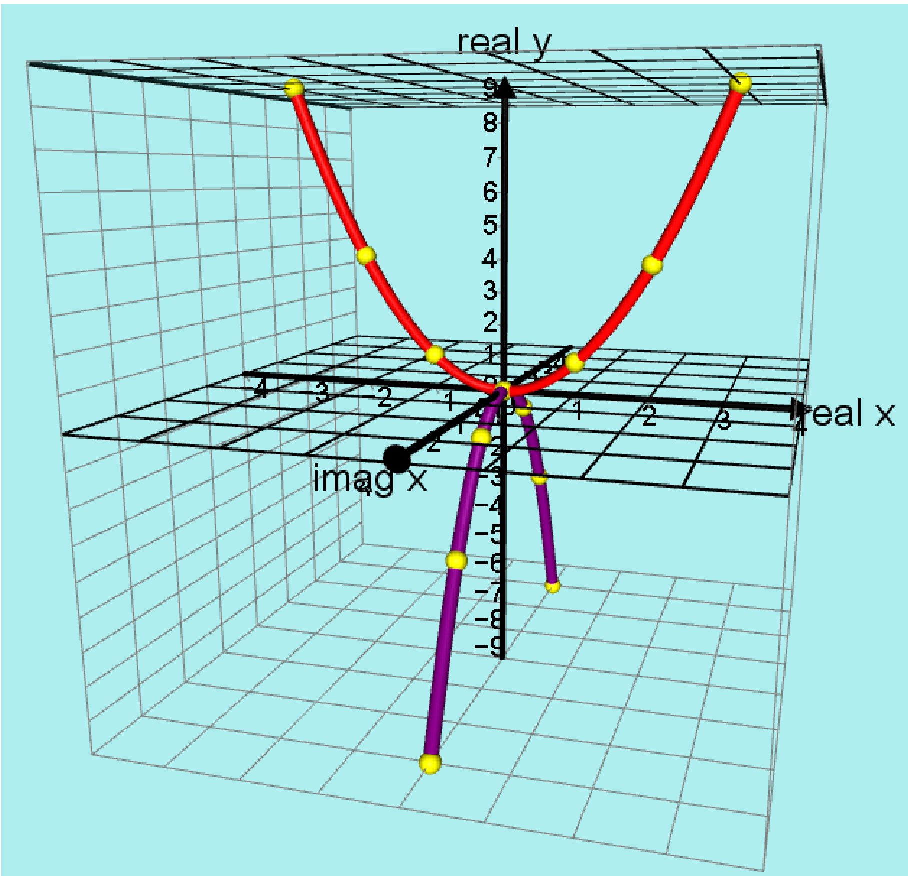
We appear to have 4 intersections at (1, 1), (– 1, 1), (***i***√2, – 2) and (–***i***√2, – 2)

I will explain how such points can be on both these graphs by considering each one separately.

If $\mathbf{y = x^2}$ the usual points we use are…
x = 0, y = 0
x = ±1, y = 1
x = ±2, y = 4
x = ±3, y = 9

…but we can choose some imaginary x values which produce REAL y values!
x = ±i, y = –1
x = ±2i, y = –4
x = ±3i, y = –9

To put these points on a graph we need an extra x axis for these imaginary values. This now produces another parabola underneath the basic $y = x^2$ but at right angles to it as shown below…



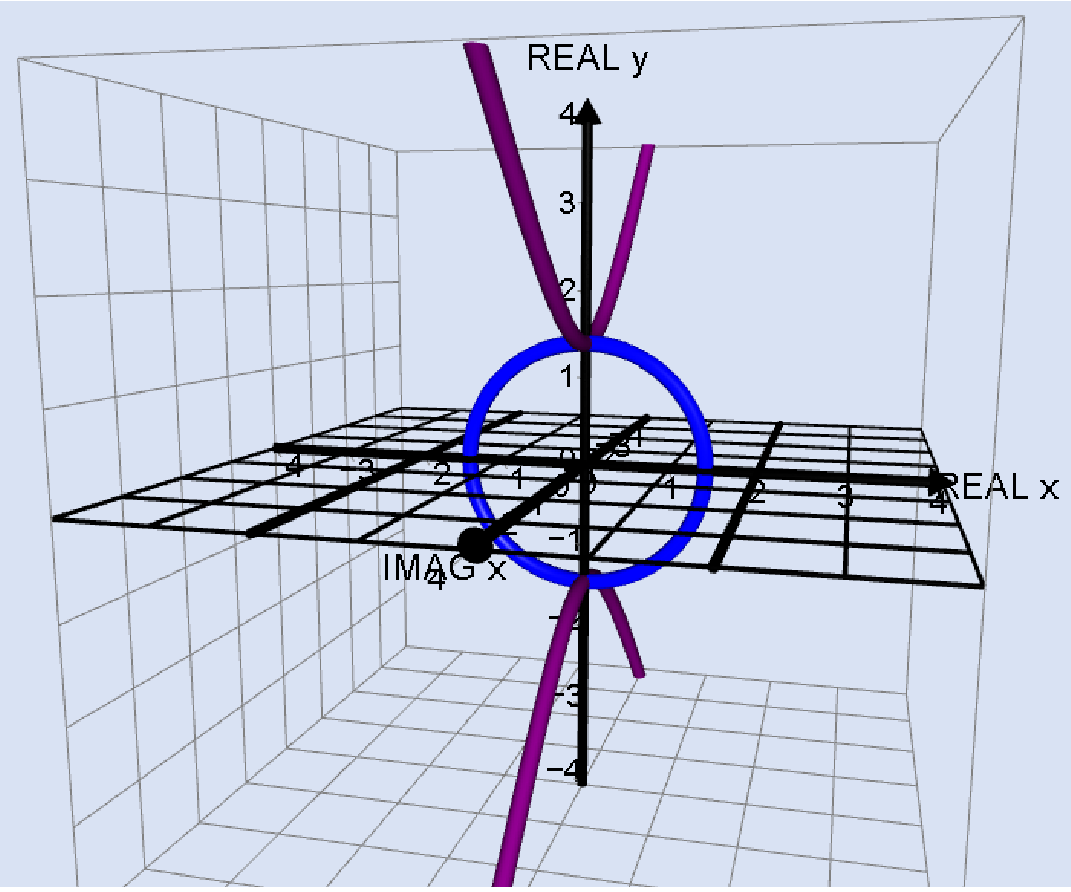
If $\mathbf{y^2 + x^2 = 2}$ we can't find a lot of integer points other than (±1, 1) and (±1, –1)

If x = 0 we get y = ±√2 and if y = 0 we get x = ±√2
and using all these points this is a circle of radius √2.

But we can choose some real y values that require imaginary x values…

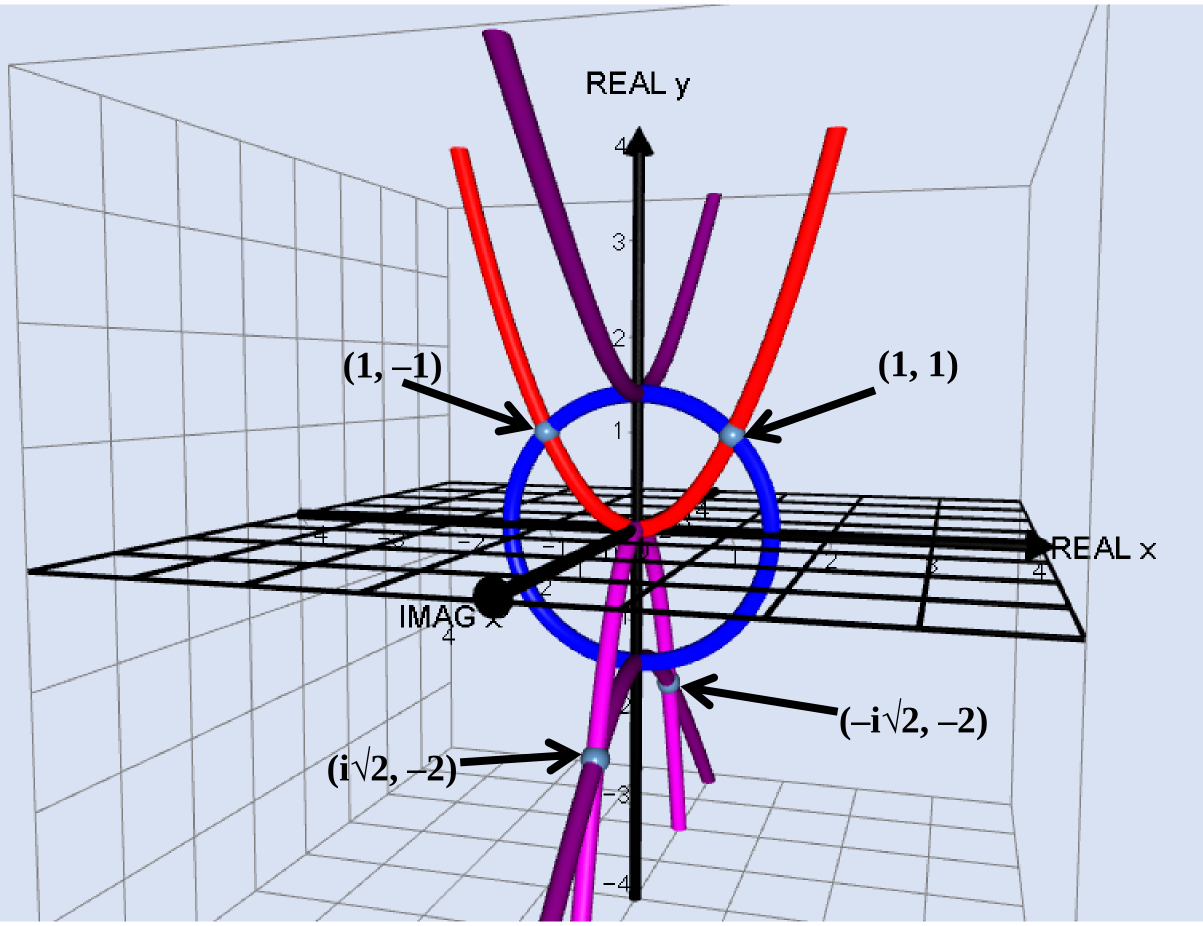
If y = 2 then x^2 = **–2** and **x = ±i√2**
If y = 3 then x^2 = **–7** and **x = ±i√7**

These points form a **hyperbola** as shown below…



Now if I put both graphs together we can see that there are 4 intersection points as predicted by the algebra shown above!

Graph showing the intersection points of $y = x^2$ and $y^2 + x^2 = 2$



You can find out more about these fascinating graphs on my website…

www.phantomgraphs.weebly.com

I will show how I worked out the equations of the phantom graphs for the above problem.

The basic 2D version of **y** = **x**^{2} just has real x values and real y values.
I will allow those complex x values which still produce real y values when substituted into the graph's equation. I will replace $\boldsymbol{x}$ with $\boldsymbol{x} + \boldsymbol{iz}$

The equation $\boldsymbol{y = x^2}$
becomes $\boldsymbol{y = (x + iz)^2}$
expanding $\boldsymbol{y = x^2 - z^2 + 2ixz}$ **----------Equ 1**

I only want REAL values of y so $\boldsymbol{2ixz}$ has to be zero!

This means $\boldsymbol{z = 0 \text{ or } x = 0}$

Subs $\boldsymbol{z = 0}$ in **Equ 1**	Subs $\boldsymbol{x = 0}$ in **Equ 1**
$\boldsymbol{y = x^2}$	$\boldsymbol{y = -z^2}$
This is the equation of the basic parabola in the x, y plane.	This is the equation of the ***phantom*** parabola in the **y, z** plane at right angles to the basic parabola and underneath it

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The equation $\boldsymbol{y^2 + x^2 = 2}$
becomes $\boldsymbol{y^2 + (x + iz)^2 = 2}$
expanding $\boldsymbol{y^2 + x^2 - z^2 + 2ixz = 2}$ **----------Equ 2**

Again, I only want REAL values of y so $\boldsymbol{2ixz}$ has to be zero!

This means $\boldsymbol{z = 0 \text{ or } x = 0}$

Subs $\boldsymbol{z = 0}$ in **Equ 2**	Subs $\boldsymbol{x = 0}$ in **Equ 2**
$\boldsymbol{y^2 + x^2 = 2}$	$\boldsymbol{y^2 - z^2 = 2}$
This is the equation of the basic circle in the x, y plane.	This is the equation of the ***phantom*** hyperbola in the **y, z** plane at right angles to the basic circle.